Mass and mixing angle predictions from infra-red fixed-points

A recollection for GrahamFest

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Introduction

- Historical ramblings
- SU(5)
- SM: fixed point for the top mass; CKM matrix; Higgs mass
- Fourth generation; other models
- More recent ramblings

Vague recollections of starting project – possibly wrong. . .

- Graham took me on as a student just after he came to Oxford.
- First task – verify one-loop beta function in QCD. Ok.
- Started loop corrections to $M_Z \cos\theta / M_W = 1$ – not completed.
- Seminar on GUTs by Cecilia Jarlskog(?) in 1979.
  - Post-seminar discussion on desert(s) between $M_{\rm W}$, $M_{\rm GUT}$ and $M_{\rm P}$.
  - Does perturbation theory in GUT hold all the way up to $M_{\rm P}$?
  - Is Georgi-Glashow $SU(5)$ GUT asymptotically free?
  - Graham's plan:
    - ★ Construct asymptotically-free GUT.
    - ★ Work out consequences - new relations between fermion masses? Bottom quark known; top not yet discovered.
    - ★ Higgs mass?
    - ★ Write paper.

Basics of Georgi-Glashow $SU(5)$ model

- Each generation has fermion fields $\theta \in \overline{5}$ and $\psi \in 10$.
  - Break $SU(5) \to SU(3) \otimes SU(2) \otimes U(1)_{\mathrm{Y}}$ at M_{GUT} with adjoint Higgs.
  - Break $SU(3) \otimes SU(2) \otimes U(1)_{\mathrm{Y}} \to U(1)_{\mathrm{EM}}$ with fundamental Higgs, $\phi \in 5$, at electroweak scale, M_{EW}.
- Restrict discussion to third-generation femions: tau, bottom, top
  - Fermion masses from Yukawa interactions: $f\overline{\theta}\psi\phi$ and $h\psi^c\psi\phi$
    - ⋆ $b, \tau \in \{\theta, \psi\}$, find $m_b, m_\tau \propto f\langle\phi\rangle$, independent of h. Prediction is $m_b = m_\tau$.

      This mass relation applies at M_{GUT}, RG flow (mostly QCD) leads to "plausible" value for m_b/m_τ at M_{EW}.
    - ⋆ $t \in \psi$ (and ψ^c) only, so m_t proportional to $h\langle\phi\rangle$, independent of f. Hence m_t is independent of m_b, m_τ in this model

Can we determine Yukawas?

Graham's ideas:

1. If we demand/construct asymptotically-free (AF) GUT, can we predict m_t/m_b and Higgs mass(es)?
2. Forget asymptotic freedom; if f, h *arbitrary* at M_P, can we predict *ratio* f/h at $M_{\rm GUT}$ from RG flow towards the "infra-red"?

The plan:

- Calculate renormalisation of Yukawa couplings at one-loop order, and hence renormalisation-group equations (RGEs) for f, h.
- Requires all one-loop diagrams with two external fermions, ψ and θ (or ψ^c), and one external scalar, ϕ.
- Need group-theory factors in $SU(5)$. "*Look in Keith Ellis' thesis.*"

RGEs for Yukawas in $SU(5)$

Define renormalisation scale μ, and $t \equiv \log(\mu/\mu_0)$. RGEs are coupled:

$$
\begin{aligned}
16\pi^2 \frac{dg}{dt} &= -\frac{b}{2} g^3 \\
16\pi^2 \frac{df}{dt} &= f \left[Af^2 + Bh^2 - Cg^2 \right] \\
16\pi^2 \frac{dh}{dt} &= h \left[Df^2 + Eh^2 - Fg^2 \right]
\end{aligned}
$$

with $b/2 = 40/3$, $A = 7$, $B = -3/2$, $C = 18$, $E = 1$, $F = 108/5$.

Admission: A is (slightly) incorrect because I thought one diagram vanished. It doesn't (Machacek & Vaughn).

RGEs for gauge coupling in $SU(5)$

- Solution for gauge coupling is well known:

$$g^2(\mu) = \frac{g^2(\mu_0)}{1 + \frac{b}{16\pi^2} g^2(\mu_0) \log(\mu/\mu_0)}$$

- Asymptotically free for $b > 0$: $g^2(\mu)/g^2(\mu_0) \to 0$ as $\log(\mu/\mu_0) \to \infty$. But it runs logarithmically *s l o w l y* .
- Don Perkins in first lecture of graduate course on strong interactions: "α_S doesn't run, it doesn't even walk, it crawls. . ."

RGEs for scaled Yukawas in $SU(5)$

- Define new variables $\overline{h} = h/g$, $\overline{f} = f/g$ *(CEL)*. RGEs for ratios:

$$8\pi^2 \frac{d\overline{f}^2}{dt} = g^2\overline{f}^2 \left[A\overline{f}^2 + B\overline{h}^2 - C + b/2\right]$$

$$8\pi^2 \frac{d\overline{h}^2}{dt} = g^2\overline{h}^2 \left[D\overline{h}^2 + E\overline{f}^2 - F + b/2\right]$$

- Get fixed points when RHSs are zero. Stability matrix tells us:
  - UV stable fixed points at $\overline{f}^2 = \overline{h}^2 = 0$. Yukawas flow towards *zero* at large mass scales $\mu \gg \mu_0$.
  - IR stable fixed points at $\overline{f}^2 = 0.93$, $\overline{h}^2 = 1.22$. Yukawas flow towards $f^2 = 0.93g^2$, $h^2 = 1.22g^2$ at low mass scales $\mu \ll \mu_0$.
  - Two mixed-stability fixed points with one Yukawa equal to zero.

Fixed Points for Yukawas in $SU(5)$

- Properties of Fixed Points
  - IRSFP is at $f^2 \approx h^2 \approx g^2$, which gives $m_t \approx m_b$.
    Lower bound on m_t from PETRA was $O(15\text{ GeV})$ at the time.
  - At fixed point $m_t \approx m_b \approx O(200\text{ GeV})$ – from calculated values of $M_{\rm GUT}$ and RG flow for QCD coupling g^2.
  - If assume h, f are *not* close to fixed-point values at M_P:
    - ⋆ Is there enough "phase space" between $\mu_0 = M_P \approx 10^{19}\text{GeV}$ and $\mu = M_{\rm GUT} \approx 10^{15}\text{GeV}$ for couplings to be swept towards fixed points?
    - ⋆ Logarithmic flow $+$ Numerical simulation $\rightarrow$ *No!*
- Conclude: $SU(5)$ IRSFP not applicable to m_t/m_b ...

RGEs for Yukawas in Standard Model

Graham's next idea:

- Assume GUT exists, evaluate RG flow for Yukawas between M_{GUT} and M_{EW}.
- Unbroken "effective" gauge group is $SU(3) \otimes SU(2) \otimes U(1)_{\mathrm{Y}}$ throughout this region. RG flows described by Standard Model RGEs, independent of GUT gauge group.
- Can we predict m_t from IRSFP(s)? There is much more "phase space" between M_{GUT} and M_{EW}, so fixed point(s) may be approached more closely.

RGEs for top-quark Yukawa in Standard Model

Simplest model:

- Since Yukawas are proportional to fermion mass, ignore all but the top-quark Yukawa h_t and the QCD coupling g_3.
- The RGEs are:

$$\begin{aligned} 16\pi^2 \frac{dg_3}{dt} &= -\frac{b_3}{2} g_3^3 \\ 16\pi^2 \frac{dh_t}{dt} &= h_t \left[A h_t^2 - B g_3^2 \right] \end{aligned}$$

with $b/2 = 11 - 2/3 n_f = 7$, $A = 9/2$, $B = 8$.

RGE flow for top Yukawa in Standard Model

- Scaling $\overline{h}_t \equiv h_t/g_3$, gives

$$8\pi^2 \, \frac{d\overline{h}_t^2}{dt} = g_3^2 \, \overline{h}_t^2 \left[A\overline{h}_t^2 - B + b/2 \right]$$

- Can solve this for $\overline{h}_t^2$ (and hence h_t^2):

$$h^2(\mu) = h^2(\mu_0) \, \frac{\left(\frac{g_3^2(\mu)}{g_3^2(\mu_0)}\right)^{2B/b}}{1 + \frac{A}{B-b/2} \left(\frac{h_t(\mu_0)}{g_3(\mu_0)}\right)^2 \left[\left(\frac{g_3^2(\mu)}{g_3^2(\mu_0)}\right)^{2B/b-1} - 1 \right]}$$

with $b/2 = 11 - 2/3 n_f = 7$, $A = 9/2$, $B = 8$.

RGE flow for top Yukawa in Standard Model

- Putting in the numbers

$$8\pi^2 \frac{d\overline{h}_t^2}{dt} = g_3^2\, \overline{h}_t^2 \left[(9/2)\overline{h}_t^2 - 8 + 7 \right]$$

- Can solve this for $\overline{h}_t^2$ (and hence h_t^2):

$$h^2(\mu) = h^2(\mu_0)\, \frac{\left(\frac{g_3^2(\mu)}{g_3^2(\mu_0)}\right)^{8/7}}{1 + \frac{9}{2}\left(\frac{h_t(\mu_0)}{g_3(\mu_0)}\right)^2 \left[\left(\frac{g_3^2(\mu)}{g_3^2(\mu_0)}\right)^{1/7} - 1 \right]}$$

  - $\overline{h}_t^2$ (and hence h_t) has an UVSTP at $\overline{h}_t^2 = 0$.
  - $\overline{h}_t^2$ has an IRSFP at $\overline{h}_t^2 = 2/9 g_3^2$, as we'd hoped – PR fixed point. (Wikipedia)

Rate of approach to the fixed point

How quickly is the fixed point approached?

- The 1980 value for $g_3^2(M_{\rm EW})$ gave $m_t \approx 110\text{GeV}$ – much bigger than (almost) everyone else was predicting or expecting at the time.
- Estimating corrections from EW couplings gave $m_t \approx 135\text{GeV}$. I'm still not quite sure how Graham got this - numerical integration suggested a slightly different result.
- In order to approach the fixed point quickly, we need $B \gg b/2$. Unfortunately, 8 is not sufficiently greater than 7 to drive an arbitrary h_t very close to the fixed point, even from $M_{\rm GUT} \to M_{\rm EW}$.
- Chris Hill (1981) introduced the effective (μ dependent) fixed point, which gave $m_t \approx 240$ GeV, and is approached much more rapidly.

Including other fermions and the Higgs

- Including all the fermions of the Standard Model, after SSB, in mass eigenstate basis:

$$\mathcal{L}_{\text{Yukawa}} = \left(\overline{u}_L M_u u_R + \overline{d}_L M_d d_R\right)\left(1 + \phi^0/v\right) + \left(\overline{u}_L U_C M_d d_R - \overline{u}_R M_u U_C u_L\right)\phi^+/v + \text{hc}$$

- u, d are 3-vectors of $Q = 2/3$, $Q = -1/3$ quark fields, M_u, M_d are (diagonal) mass matrices, ϕ_0 is (complex) Higgs, ϕ^+ is Goldstone boson eaten by W^+, and U_C is CKM matrix
- RGEs become matrix ODEs. Graham's initial calculation was in weak eigenstate basis for 4 flavours, $U_C = U_u U_d^\dagger$. Gets too messy with 6 flavours – due to multiple θ_i, δ, and fermion phase transformations.

IRSFPs for CKM mixing angles and phase

- Rediagonalise M_u, M_d when let $\mu \to \mu + \delta\mu$, so U_C has RG flow.
- Make μ-dependent phase transformations on fermion fields such that first row and column of U_C remain real at all scales μ.
- Disentangle RGEs for individual generalised Cabibbo angles θ_i, $i = 1 \ldots 3$ and CP-violating phase δ.
- Find θ_i RG flow is not affected by phase transformations, RG flow of δ *is* affected – intuitively obvious?

  (Ma & Pakvasa performed similar RG analysis, but without phase transformations.)
- Find θ_i, δ RG flow is entirely due to Yukawa couplings – longitudinal modes of W; EW gauge couplings cancel in RGEs.

IRSFPs for CKM mixing angles and phase

- Keeping only h_t, defining $s_i \equiv \sin\theta_i$, gives, for example:

$$
\begin{aligned}
16\pi^2 \frac{d\theta_1}{dt} &= \frac{3}{2} h_t^2 s_1 c_1 s_2^2 \\
16\pi^2 \frac{d\delta}{dt} &= 3 \sin\delta \left(c_1 s_2 c_2 s_3 / c_3 \right) h_t^2
\end{aligned}
$$

- All θ_i and δ have IRSFPs at zero – which are approached rapidly only for large m_t. Numerical results for $m_t = 173\mathrm{GeV}$? (Code lost)
- Approximate solutions for $\theta_i(\mu)$, $\delta(\mu)$ obtained during month-long visit to CERN/Annecy over Easter of 3rd year – made possible by Graham.
- Graham lent me his skis. I arrived safely at Geneva airport; the skis didn't. . . *Panic!*

IRSFPs for Higgs mass(es)?

- *SU*(5): I can't remember what I/we did – it's not in my thesis. . .
- Standard Model: RHS of RGE for Higgs self-coupling λ depends on products of λ, EW gauge couplings g, g', and on h_t^2 – which is not known *a priori*.
- Simplifications for FP analysis: assume top Yukawa is at fixed point $h_t^2 = 2/9 g_3^2$. Ignore EW couplings. Scale $\lambda = \overline{\lambda} h_t^2$, get RGE

$$16\pi^2 \frac{d\overline{\lambda}}{dt} = \frac{2}{9} g_3^4 \left[4\overline{\lambda}^2 + 75\overline{\lambda} - 36 \right]$$

- IRSFP at $\overline{\lambda} = 0.47$, which is approached rapidly - but only useful if h_t is close to its fixed point – seemed unlikely.
- IRSFP $\rightarrow m_H \approx 72$ GeV. Numerical integration for wide range of heavy m_t gave $m_H = (50 \rightarrow 100)$ GeV – assuming desert/GUT.

Fourth Generation

- In 1980, could have had $m_t < 20$ GeV. Could there be a heavy fourth generation (T, B)? (With a not-light ν.)
- RGEs: keeping Yukawas for T & B only:
  - Previous $SU(5)$ analysis unchanged, IRSFP at $m_T \approx 1.15 m_B$; IR region is $\mu \approx M_{\rm GUT}$; FP approached slowly.
  - RGEs in 4-generation SM:

$$16\pi^2 \frac{dh_B}{dt} = h_B \left[3/2\left(h_B^2 - h_T^2 \left|U_{\rm TB}\right|^2\right) + 3(h_B^2 + h_T^2) - 8g_3^2\right]$$

$$16\pi^2 \frac{dh_T}{dt} = h_T \left[3/2\left(h_T^2 - h_B^2 \left|U_{\rm TB}\right|^2\right) + 3(h_B^2 + h_T^2) - 8g_3^2\right]$$

$$16\pi^2 \frac{d}{dt}\left(\frac{h_T}{h_B}\right) = \frac{h_T}{h_B}\left[3/2\left(h_T^2 - h_B^2\right)\left(1 + \left|U_{\rm TB}\right|^2\right)\right]$$

  - IRSFP at $h_T^2 = h_B^2 = (7/18) g_3^2 \to m_T = m_B \approx 150$ GeV.
  - Rate of approach to FP much(?) faster than 3-generation case.

Other models

- Also studied L-R symmetric models with multiple Higgs.
- Higgs with small vevs give mass to fermions $\rightarrow$ light fermions have larger Yukawas $\rightarrow$ IRSFPs relevant.
- Higgs with larger (EW scale) vevs give masses to weak bosons.
- Reading it now $\rightarrow$ very imaginative!
- Two-generation model: Graham drafted paper - I lost the draft.
- It was recovered some years later in in-laws' outdoor store room - stored safely when I was a postdoc in Santa Barbara.
- Others had done similar things by then, including people at UCSB.

Where are they now?

Fellow students:

- Simon Duane:
  - NPL since 1987; Acoustics and Ionising Radiation.
  - Worked with Simon 1985-87 on Lattice Field Theory$\rightarrow$ HMC.
  - Postcard from Siberia.
- Sean Monaghan: Computer Science & Electrical Engineering at Essex.
- George Christos: (Ex) Applied Maths at Curtin University, Perth, WA
- John Wheater – you know. . .
- Caroline Fraser and Elizabeth Gardner, sadly no longer with us - but not forgotten.
- Ken Parker, Jack McGinley, Tim Robinson, Arthur Maciel, RM Doria, – don't know . . .
- Maggie(?) (secretary) – bumped into at her UC San Diego mid '80s.

Afterwards

- Graham became supersymmetric;
- I became discrete. . .

There was another Graham effect:

- Two years after the month at CERN, I learned to ski properly. To prove it, I bumped into Graham (more-or-less literally) on a ski slope at Alpe d'Huez some years later.

Finally. . .

- Thanks for everything, Graham. You were a great supervisor.
- Have a long (and active) retirement! Ancient theses may not help, but whisky might. . .